Changes in Students' Mathematical Discourse When Describing a Square

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Developing students' geometric reasoning skills is dependent on the quality of task designs and the role of the teacher. The purpose of this study was to apply Sfard's (2008) interpretive framework to analyse changes in students' mathematical discourse. This paper reports on the results of an investigation into the ways one class of Year 7 students communicated their understanding of a square. The results showed that students grappled with the necessary elements involved with describing a square leading to several misconceptions about its key attributes, and raises questions about task designs and the teacher's role in developing geometric reasoning.

The current educational interest in science, technology, engineering, the arts and mathematics [STEAM] in Australia presents opportunities for richer connections of mathematics with other learning areas. Geometry is a significant strand of mathematics as it can be applied across mathematics and to other disciplines. Geometry helps develop students' spatial reasoning skills and abilities to solve real-world problems (Marchis, 2012). Lowrie, Logan and Ramful (2016) found a strong relationship between students' spatial reasoning and mathematics performance, highlighting the importance of promoting spatial reasoning in the Australian Curriculum.

Research indicates that many students have difficulties engaging in tasks that require visual, logical, and deductive thought due to a lack of spatial and geometric reasoning ability (Marchis, 2012; Oberdorf & Taylor-Cox, 1999). Commonly, students experience difficulties recognising geometrical shapes in non-standard orientation and formulating accurate definitions (Marchis, 2012), due to a lack of exposure to geometric vocabulary (Oberdorf & Taylor-Cox, 1999). Equally concerning is that teachers often retain the same misconceptions and misunderstandings of geometric concepts from their own schooling (Cunningham & Roberts, 2010; Fujita & Jones, 2006; Marchis, 2012), unaware of their own students' difficulties (Canturk-Gunhan & Cetingoz, 2013), and making it unlikely that they would provide learning experiences for extending their students' geometric reasoning.

Reasoning with Shapes

Geometry begins with perception and imagery - an ability to visualise with a 'picture in the mind' (Clements, 1982). Visualisation is vital for communicating geometric concepts both verbally and non-verbally at all levels of geometric reasoning (Battista, 2001). Visualisation involves generating a mental image, whether static or dynamic, and understanding that an image depicts visual or spatial information (Presmeg, 2006). Visualisation, therefore, is a complex process involving imagery, with or without a diagram, to organise information into meaningful structures that are important in guiding the analytical development of a solution to geometric problems (Fischbein, 1993).

Geometric reasoning develops from processes of recognising and manipulating mental objects and the relations among those objects (Lowrie, Logan & Ramful, 2016). In geometric reasoning, what is important is to have a sense that because a shape has certain

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properties, other properties must also be true. It is important for students to be able to deduce facts by interpreting the geometric information that they ‘see’ in their minds (Fujita & Jones, 2006). A specific geometric diagram embodies the attributes of a class, providing students with prototypes. Prototypes in geometry are generalised representations having common visual characteristics and are useful for simple manipulations. However, prototypes are limited references to geometrical concepts having internal constraints of organisation and do not support hierarchical, inclusive definitions (Presmeg, 2006). Students need to be able to explore shapes by ‘seeing the parts’ – a notion that Owens (2003) referred to as *disembedding*. An image is no longer a ‘picture in the mind’ but rather images are abstract, malleable, less crisp, and are often segmented into parts.

Diagrams are an essential component of geometric reasoning (Dreyfus, 1991). The effective use of diagrams as a communicative tool for high school students necessitates an understanding of the universal mathematical signifiers used to indicate particular properties on a figure (such as a square in a corner for a right angle, or the use of arrowheads for parallel lines). Diagrams are as powerful as definitions (Tall & Vinner, 1981). However, students prefer to rely on visual prototypes rather than verbal definitions when identifying and classifying shapes as they typically remember prior experiences with diagrams presented by their teachers (Cunningham & Roberts, 2010).

Definitions serve the dual role of identifying a category to which a shape belongs, and indicating how it might be distinguished from other objects in that category. *Concept definitions* are word formations used to specify that concept, and a *concept image* is the total cognitive structure that is associated with the concept (Fujita & Jones, 2006), including all the mental pictures and associated properties and processes (Tall & Vinner, 1981). Fischbein (1993) defined the notion of a *figural concept* – a square, for example, is a concept as well as a geometric figure. Many secondary teachers expect a one-way process for concept formation, that is, “…the concept image will be formed by means of the concept definition” (Vinner, 1991, p. 71). Consequently, their students tend to use partitional definitions creating difficulties with logically connecting ‘new’ information with what they have been previously taught.

Sfard’s Interpretive Framework for Mathematical Discourse

The discourse used in the classroom has a significant influence on what and how students learn mathematics (Ferreira & Presmeg, 2004). Analysis of student discourse is an important aspect in understanding students’ interpretations of tasks, as well as their ability to communicate geometric concepts (Berenger, Barkatsis, Seah, 2017). According to Sfard (2008), mathematical discourse is exhibited by four inter-related components. These are:

- *Keywords* – Shapes are described and defined in distinctly mathematical ways. How a shape is seen and interpreted by a student is revealed by their use of *keywords*.
- *Visual mediators* –As part of the communication process that helps define shapes and their properties, visual objects that are operated on are known as *visual mediators*.
- *Narratives* – A sequence of expressions or statements used to frame descriptions of objects, either spoken or written, are known as *narratives*. Narratives are subject to rejection or acceptance as deductive accounts of an endorsed consensus.
- *Routines* – Specific repetitive patterns characteristic of creating and substantiating narratives about shapes form *routines* of mathematical discourse.

Mathematics discourse is made distinct by the tools of *keywords* and *visual mediators* giving rise to *narratives* and possible *routines* one applies to shared practices of reasoning, arguing, and symbolising while communicating particular mathematical ideas (Cobb, Stephan, McClain & Gravemeijer, 2010). Convincing others through a common discourse is a necessary component in the meaning-making process of geometry (Berenger, Barkatsis, Seah, 2017). Students need to be able to connect learned facts to construct logical arguments as endorsed mathematical discourse (Sfard, 2008). Conversely, student misconceptions are revealed by difficulties in formulating mathematically acceptable descriptions or definitions. Their narratives are therefore subject to rejection.

Sinclair and Yurita's (2008) application of Sfard's interpretive framework with secondary teachers working in a dynamic geometric environment [DGE] revealed changes in their use of *visual mediators* and *narratives* to perceive and reason about mathematical objects with their students. Few studies, however, have used this framework to analyse students' reasoning with geometric concepts. One study by Seah, Horne and Berenger (2016) found that middle year students had limited ability to use *keywords* to formulate accurate and complete *narratives* such as definitions. In a related study, Berenger, Barkatsis and Seah (2017) found that Year 8 students experienced difficulties aligning *keywords* and *narratives* to *visual mediators* when describing 2-dimensional shapes.

Method

Students in one Year 7 class in an inner suburban secondary school in Melbourne were given two written tasks. **Task A** asked students *What is a square?* The teacher instructed them to record as much as they knew, to work individually, and did not allow discussion before or during the task. Questions such as *"can we draw a picture?"* were not allowed as a means of ensuring that students did not prompt each other through questioning.

After responses had been collected, the teacher conducted a 30-minute teaching episode to assess current student thinking and reinforce mathematical concepts drawn out by the task. This session allowed students to state known facts about a square as the teacher listed them on the whiteboard. She drew several squares and labeled geometric properties according to student responses. The teaching episode was recorded to assist the analysis of the discursive features of the teacher's communication. To assess students' retention of key ideas explored in the teaching episode, **Task B** was conducted one week later requiring students to draw a square and list its properties.

The purpose of both tasks was to understand how students in Year 7 think and communicate about, what the researcher anticipated as, a familiar geometric shape. The teacher assessed students' use of *keywords* and categorised them according to definitional properties, transformational relationships, formal property-based reasoning, and hierarchical properties in relation to a square. In this study, Sfard's interpretive framework was used to analyse changes in students' mathematical discourse about the square concept. Analysis of students' written discourse considered their use of *keywords, visual mediators* and *narratives* providing the basis for what they knew and communicated about a square, as well as what they learned about a square as a result of the teaching episode.

Results

The results from both tasks are grouped and reported together in terms of *keywords, visual mediators,* and *narratives* used by students. Student misconceptions about squares are also reported. Segments of the teaching episode that occurred between tasks are presented indicating some of the teacher's actions impacting on student learning. It is not possible to report on *routines* requiring well-defined discourse patterns over time.

Keywords

Michelle's response is representative of the way most students listed known facts about a square identifying 4 sides, 4 corners, and other properties, but without reference to right angles or use of any visual mediation (see Figure 1).

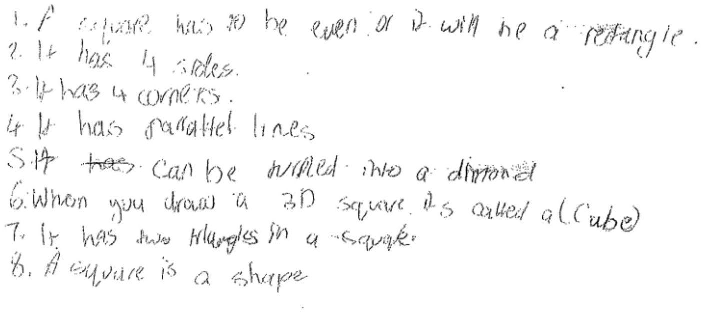

1. A square has to be even or it will be a rectangle.
2. It has 4 sides.
3. It has 4 corners.
4. It has parallel lines
5. It ~~has~~ can be turned into a dimond
6. When you draw a 3D square it's called a (Cube)
7. It has two triangles in a square.
8. A square is a shape

Figure 1. Michelle's response to Task A.

In Task A, initial analysis of keywords showed that 25% of students specified 4 sides of equal length, and 10% specified right angles. Only 5% of students provided both conditions for a square. After the teaching episode, in Task B, 29.4% of students stated the two necessary conditions for a square. There was also an increase in the use of 'new' terms of parallel and symmetry. These results are indicated in Table 1.

Table 1
Keywords Used to Describe a Square

Category of keywords	% of responses before teaching episode (Task A)	% of responses after teaching episode (Task B)
Definitional		
2D	70.0	29.4
4 sides	55.0	41.2
4 lines	20.0	17.6
4 edges	10.0	11.8
4 equal sides	25.0	29.4
even sides	10.0	11.8
4 corners	80.0	100
4 right angles (90^o)	10.0	29.4
Transformational		
symmetry	5.0	35.3
Formal property-based reasoning		
parallel	5.0	29.4
Hierarchical		
rectangle	10.0	0
Other		
Personal (eg. dice, grids)	45.0	0
3D reference to cubes	10.0	0

Further examination of the changes in keyword usage indicated a decrease in the proportion of students referring to a square as being 2-dimensional, and a decline in personal references (ie. dice, cubes). These changes may be due to the structure of Task B asking students to draw a square before listing its properties, or as a result of the teaching episode after Task A where some of the critical attributes of a square were highlighted.

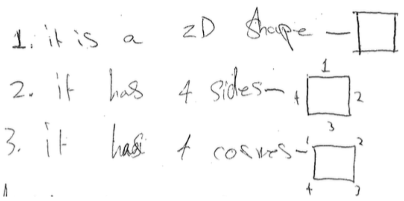
Visual Mediators

Table 2 shows that before the teaching episode, no students indicated the necessary and sufficient properties of a square on a diagram. General shape outlines were produced by 25% of students where they used personal signifiers of arrows and numbering to indicate sides of equal length (see Figure 2). After the teaching episode, only 5.9% of students used correct mathematical signifiers when depicting a square. Diagrams used by students did not always match their accompanying narratives (see Figure 3).

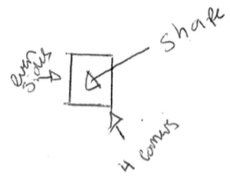
Table 2
Visual Mediators for a Square

Type of visual mediator	% of responses before teaching episode (Task A)	% of responses after teaching episode (Task B)
No diagram	65.0	0
Incorrect diagram	10.0	0
General shape (no signifiers)	25.0	88.2
Right angle signifiers	0	5.9
Equal side signifiers	0	0
Both angle and side signifiers	0	5.9

In the first instance it was not automatic for students to include diagrams nor was it seen as necessary when describing a square. The lack of accurate diagrams of squares with signifiers after the teaching episode indicated an ongoing problem with students' use of visual mediators to indicate key geometric properties other than its general shape.

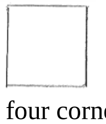


Fadi: It has right angles in its corners and it has one face

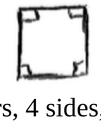


Mary: That a square has 4 even sides. It has 4 corners. It's a shape, can be 3D or 2D

Figure 2. Sample of visual mediators and personal signifiers used to describe a square and accompanying narratives before teaching episode (Task A).



Sharee: It has four corners. All corners are about a 90º angle and a square also has a straight angle. It has 4 sides and has parallel lines. All sides are equal length.



Abdul: 4 corners, 4 sides, 4 90º angles

Figure 3. Sample of visual mediators used to describe a square and accompanying narratives after teaching episode (Task B).

After the teaching episode, almost every student used rulers to produce neat diagrams of squares. However, students were unable to retain information conveyed to them about using diagrams to communicate geometric properties such as equal sides and right angles even if they accurately listed the necessary properties of a square. Instead, students had retained the importance of neatness emphasized during the teaching episode.

Narratives

Analysis of written narratives revealed imprecise thinking about squares. The types of misconceptions recorded as shown in Table 3 indicated a large proportion of students made reference to a 3-dimensional object (cube or box) despite having also referred to it as a quadrilateral or 2-dimensional shape. Other misconceptions relate mainly to orientation.

Table 3
Students' Misconceptions when Describing a Square

Common misconceptions	% of responses
2-dimensional version of a cube, box, 6 faces	45.0
is three-quarters of an A4 page	15.0
made up of 2 triangles	5.0
rotated becomes a diamond	5.0
stretched to become a rectangle	10.0
vertical and horizontal	5.0
Other	15.0

Michelle incorrectly stated that a square "…can be turned into a diamond…a 3D square is called a cube…has two triangles…" Michelle's response indicated multiple misconceptions about a square, and was also detected in the work of three other students.

The Teaching Episode

The teaching episode conducted between Task A and Task B provided an opportunity for the teacher to assess student understanding of geometric concepts, and to emphasise keywords and the significance of diagrams. The teacher supported student responses through her questioning to draw out descriptions from students (see Figure 4).

Student A: [a square] has a right angle
Teacher: how would I show it here? (*invited the student to add to the diagram on the board*)
Teacher: what does a right angle mean? (*drawing out further meaning*)

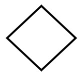
Figure 4. Dialogue from teaching episode for indicating a right angle.

The teacher folded a square piece of paper to help define *diagonal* and *symmetry* concepts. Students were encouraged to use hand gestures to connect ideas of horizontal and vertical symmetry and parallel lines. Gestural forms of communication are relevant to the discourse narrative (Ferreira & Presmeg, 2004; Sfard, 2008) (see Figure 5).

Student B: A square has parallel lines.
Teacher: What does that mean?
Student B: …go in the same direction… (uses his hands to motion movement in one direction)

Figure 5. Dialogue from teaching episode for indicating parallel lines.

The teacher's questions challenged students to provide more detailed responses thus building an exhaustive list of properties of a square. She later drew a rotated square on the board (see Figure 6) and asked the students what it was.



Teacher: So, a rhombus. A diamond. What are you telling me?
Student C: Same thing.

Figure 6. Teacher's rotated square and dialogue.

The teacher wrote *Diamond* above the object as well as *a rhombus*. The teacher's acceptance of 'same thing' indicated her own misconception in relation to orientation and conveyed to her students, that is, a diamond, a rhombus and a rotated square are all the same. Both the teacher and her students believed that the non-critical attributes of an object, such as its orientation, are important in its *concept definition*.

Discussion and Conclusion

Geometric reasoning is characterised by the use of specific keywords and visual mediators giving rise to endorsed narratives. The analysis of students' use of keywords, visual mediators and narratives indicated significant gaps in their ability to describe the necessary properties of a square and to determine what is sufficient. This claim is substantiated by only 29% of students being able to list 4 equal sides and 4 right angles yet none depicted this accurately on their diagrams despite this being modeled by the teacher.

Application of Sfard's interpretive framework revealed that students do not accept visual images as powerful aspects of geometric discourse. The use of visual mediators improved marginally after the teaching episode with most students depicting general shapes without the use of signifiers for equal sides and right angles, due partly to an over-emphasis on neatness rather than the critical attributes of a square. If, as Dreyfus (1991) and Presmeg (2006) suggested, students might generate visual images but have a basic reluctance to use them to communicate geometric concepts, then it could be conjectured that students' prior experiences of reasoning with shapes were restricted to basic recognition and memorisation activities. These activities are often characterised by listing facts without emphasis on the need for mathematically acceptable visual mediators.

This study raised several questions about geometric task designs and the role of the teacher. Students were asked *What is a square?* in Task A. Such open-ended tasks are commonplace and have merit in understanding the extent of student knowledge. However, the focus on listing facts to reason about shapes, emphasised in Task A and the teaching episode, placed weight on written narratives above visual mediation hampering students' ability to discern and articulate the minimal properties needed to describe, and therefore define, a square. Further, students may not have understood the purpose of the tasks nor found them engaging, suggesting a lack of exposure to non-routine geometric tasks. This was indicated by their inability to retain key concepts presented to them, and implying the need for newly learned concepts to be reinforced through further teaching activities.

How teachers question, listen, and respond to their students is crucial in their understanding of mathematics (Ferreira & Presmeg, 2004). This study identified several misconceptions and difficulties with geometric concepts stemming from students' imprecise or personal concept images of squares. The teaching episode indicated concerns about the teacher's content knowledge as it shed light on her own misconceptions with the square concept, and provided an explanation for how students might develop similar misconceptions in the first instance. This study was limited to a snapshot of teaching and learning in one classroom, and indicates a direction for future research into effective teaching approaches to sustain geometric reasoning.

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